Hence, substituting the values of KN and OK,

$$\frac{\tan x + \tan u}{1 - \tan x \tan u} = \frac{2 \cot z - \cos y}{\sin y}.$$

Substituting the value of $\tan u$,

$$2 \sin y(\tan z - \cot z) = 2 \tan x \cos y(\cot z + \tan z) - 5 \tan x.$$

But $\tan z - \cot z = -2 \tan y$ and $\cot z + \tan z = \dfrac{2}{\cos y}$.

Hence, substituting and reducing,

$$\tan x = 4 \tan y \sin y,$$

the required relation.

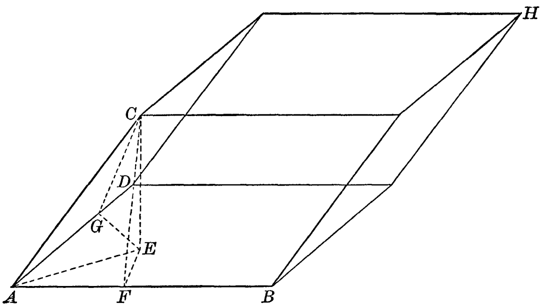
Also solved by PAUL CAPRON and J. W. CLAWSON.

458. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Given edges l, m and n of a parallelopiped and angles a, b and c which the edges make with one another. Show that, if $s = \dfrac{a + b + c}{2}$, the volume equals

$$2lmn\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$

SOLUTION BY FRANK R. MORRIS, Glendale, Cal.



Given the parallelopiped AH with $\angle BAC = a$, $\angle DAC = b$, $\angle BAD = c$, and $AB = l$, $AC = m$, $AD = n$.

The area of the base BD is equal to $ln \sin C$.

From the vertex C drop a perpendicular to the base BD meeting it at E. From E draw perpendiculars to AB and AD meeting the lines at F and G, respectively. CE is the altitude of the parallelopiped and we know that the volume is $V = CE \cdot ln \sin c$. (1).

The triangles AEC, AFC, AGC, AFE and AGE are right triangles. Hence, we have

$$CE^2 = m^2 - AE^2, \quad (2) \qquad AF = m \cos a, \ AG = m \cos b, \quad (3)$$

$$\cos \angle EAF = \frac{AF}{AE}, \quad (4) \qquad \text{and} \qquad \cos (c - \angle EAF) = \frac{AG}{AE},$$

or

$$\cos c \cos \angle EAF + \sin c \sqrt{1 - \cos^2 \angle EAF} = \frac{AG}{AE}. \tag{5}$$

E lim inating $\cos \angle EAF$ from (4) and (5)

$$\cos c \frac{AF}{AE} + \sin c \sqrt{1 - \frac{AF^2}{AE^2}} = \frac{AG}{AE}.$$

From this equation we get

$$AE^2 = \frac{AG^2 - 2AG \cdot AF \cos c + AF^2}{\sin^2 c}.$$

Substituting the values of AF and AG from (3),

$$AE^2 = \frac{m^2 \cos^2 b - 2m^2 \cos a \cos b \cos c + m^2 \cos^2 a}{\sin^2 c}.$$

Then from (2),

$$\begin{aligned}
CE^2 &= \frac{m^2}{\sin^2 c}(\sin^2 c - \cos^2 b + 2 \cos a \cos b \cos c - \cos^2 a) \\
&= \frac{m^2}{\sin^2 c}(\sin^2 c + \cos^2 c - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c - \cos^2 b \cos^2 c \\
&\qquad + 2 \cos a \cos b \cos c - \cos^2 a) \\
&= \frac{m^2}{\sin^2 c}\{(1 - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c) - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{(1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{\sin^2 b \sin^2 c - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{\sin b \sin c + (\cos a - \cos b \cos c)\}\{\sin b \sin c - (\cos a - \cos b \cos c)\} \\
&= \frac{m^2}{\sin^2 c}\{\cos a - \cos (b + c)\}\{\cos (b - c) - \cos a\} \\
&= \frac{m^2}{\sin^2 c}\left\{2 \sin\left(\frac{a+b+c}{2}\right) \sin\left(\frac{b+c-a}{2}\right)\right\}\left\{\sin\left(\frac{a-b+c}{2}\right) \sin\left(\frac{a+b-c}{2}\right)\right\} \\
&= \frac{4m^2}{\sin^2 c} \sin s \cdot \sin (s - a) \sin (s - b) \sin (s - c),
\end{aligned}$$

where

$$s = \frac{a + b + c}{2}.$$

Hence, $CE = \dfrac{2m}{\sin c}\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}$. Substituting this value of CE in (1), we have

$$v = 2lmn\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$

Also solved by George W. Hartwell, Horace Olson, J. A. Caparo, and J. W. Clawson.

CALCULUS.

370. Proposed by PAUL CAPRON, Annapolis, Maryland.

The surface of a right circular cone having the semi-vertical angle α is cut by two planes, which intersect the axis at the same point, one at right angles to the axis, the other making the angle $(90° - \beta)$ with the axis. Show that if the lateral surface of the right cone is S_1 and that of the oblique cone S_2,

$$S_2 = \sum_1^\infty T_n, \qquad \text{where} \qquad T_1 = S_1,\ T_{n+1} = T_n \times \frac{2n+1}{2n}(\tan \alpha \tan \beta)^2.$$

Solution by the Proposer.

Let the vertex of the cone be the origin, its axis the z-axis, and let the intersection of the oblique plane with the xy-plane be parallel to the y-axis. Let the two planes cut the z-axis at $(0, 0, h)$, and let the radius of the right section be a. Then,